JOURNAL OF THE
CHUNGCHEONG MATHEMATICAL SOCIETY
Volume **29**, No. 1, 2016
http://dx.doi.org/10.14403/jcms.2016.29.1.23

CHARACTERIZATIONS OF FLOWS NEAR CLOSED SETS

Ki Shik Koo*

Abstract. In this paper, we obtain close characterizations of flows near a saddle set which is one of the interesting unstable closed sets. Also, we consider some conditions that closed sets are to be stable.

As an attempt to approach some problems of stability theory in dynamical systems, behaviour of flows near closed sets are studied together with the related concepts of invariance, stability, and unstability. We can find many results for the analysis of flows near arbitrary closed sets.

The purpose of this paper is devoted to a rather deep analysis of flows in the vicinity of various closed sets. We obtain close characterizations of flows near a saddle set which is one of the interesting unstable closed sets. Also, we consider some conditions that closed sets are to be stable.

A given continuous flows (X, π) on a locally compact metric space X will be assumed throughout this paper. The symbols O, D denote, respectively, the orbit and prolongation relations. The unilateral versions of these relations carry the appropriate superscript $+$ or $-$. Also, $O^+(x)$, $O^-(x)$ are called semi-orbits of x. $\omega(x)$, $\alpha(x)$ denote, respectively, the positive and negative limit set of $x \in X$. A point x of X is positively weakly attracted to a set $M \subset X$ if positive semi-orbit of x is frequently contained in each neighborhood of M. The region of positive weak attraction is denoted by $A^+_W(M)$ and M is called positive weak attractor if $A^+_W(M)$ contains a neighborhood of M. A set $M \subset X$ is called a saddle set if there exists a neighborhood U of M such that every neighborhood V of M contains at least one point x with $O^+(x) \not\subset U$ and $O^-(x) \not\subset U$.

$U \subset X$ is a neighborhood of x provided U is an open set containing x. We denote the closure, interior, boundary and complement of a set $M \subset X$ by $\overline{M}$, M°, ∂M and $X \setminus M$, respectively.

Received November 12, 2015; Accepted January 15, 2016.
2010 Mathematics Subject Classification: Primary 54H20.
Key words and phrases: saddle set, weak attractor, unstable set, stable set.
This research was supported by the Daejeon University fund (2015).

Each of the basic properties of dynamical theory used in this paper are presented in detail in references [1,2].

THEOREM 1.1. *Let $M \subset X$ be a closed set with compact boundary which contains no semi-orbits. Then, M is positively invariant if and only if M is positively asymptotically stable.*

Proof. Assume that M is positively invariant and let U be a neighborhood of M. Let x be in M. Since $O^+(x) \subset M$ and $O^+(x) \not\subset \partial M$, there is a number $t_x \geq 0$ such that $xt_x \in M^\circ$. Thus, there is a neighborhood V_x of x with $xt_x \in V_x t_x \subset M^\circ$. Since M is positively invariant, for every nonnegative number t, $(V_x t_x)t \subset M$. On the other hand, by the continuity of π, there is a neighborhood W_x of x such that $W_x t \subset U$ for all t, $0 \leq t \leq t_x$. Let $N_x = V_x \cap W_x$. Then, for any t with $0 \leq t \leq t_x$, $N_x t \subset W_x t \subset U$ and, for any t with $t \leq t_x$, $N_x t = N_x t_x(t - t_x) \subset V_x t_x(t - t_x) \subset M \subset U$. Let $W = \bigcup\{O^+(N_x) : x \in M\}$. Then, W is a positively invariant neighborhood of M which is contained in U. Thus we see that M is positively stable. Also, let y be in W and $y \in N_z$ for some $z \in M$. Since $yt \in M$ for all $t \geq t_z$, the semi-orbit $O^+(z)$ is ultimately contained in M. This shows that $W \subset A^+(M)$ and M is an attractor. Therefore, M is positively asymptotically stable.

The converse is trivial and this completes the proof. □

COROLLARY 1.2. *Let $M \subset X$ be a positively invariant closed set with compact boundary which contains no semi-orbits. Then, M is positively asymptotically stable and $\overline{X \setminus M}$ is negatively asymptotically stable.*

Proof. This follows from the facts that M is positively invariant if and only if $\overline{X \setminus M}$ is negatively invariant and the boundaries of M and $\overline{X \setminus M}$ are same. □

THEOREM 1.3. *Let $M \subset X$ be a positively (negatively) invariant closed set with compact boundary. Then every neighborhood of M contains a point x not in M such that a semi-orbit of x is fully contained in that neighborhood.*

Proof. Let M be positively invariant. If M is positively stable, then conclusion is trivial. So, assume that M is not positively stable. Let U be a neighborhood of M. Since M is not positively stable, there is a neighborhood W of M, a sequence of points $\{x_i\}$ in $W \setminus M$ satisfying that $x_i \to x \in \partial M$ and $O^+(x_i) \not\subset W$. We may assume that $\overline{W} \subset U$ and $\overline{U} \setminus M^\circ$ is compact. For each i, define $t_i = \inf\{t > 0 | x_i t \in \partial W\}$. Then we have $x_i t_i \in \partial W$ and $x_i(0,\ t_i) \subset \overline{W} \setminus M^\circ$. Suppose $\{x_i t_i\}$

converges to z in ∂W. Let s be a positive number. Since M is positively invariant we may assume that $t_i \to \infty$. Hence, there exists an integer $N > 0$ such that $t_i - s > 0$ for all integers $i \geq N$. Then, for each $i \geq N$, we have $(x_i t_i)(-s) = x_i(t_i - s) \in x_i(0,\ t_i) \subset \overline{W} \setminus M^\circ$ and $(x_i t_i)(-s) \to z(-s) \in x_i[0,\ t_i] \subset \overline{W} \setminus M^\circ$. This shows that the negative semi-orbit of z is fully contained in U. In the case that M is negatively invariant the proof is similar. This completes the proof. $\square$

M is called isolated from closed invariant sets provided there exists a neighborhood U of M satisfying that every closed invariant subset of U is contained in M.

THEOREM 1.4. *Let $M \subset X$ be a closed set with compact boundary and be isolated from closed invariant sets. Then, M is positively stable if and only if M is positively invariant and satisfies Zubov's condition, i.e. $\alpha(X \setminus M) \cap M = \emptyset$.*

Proof. First, suppose that M is positively invariant and satisfies Zubov's condition. Let U be a neighborhood of M which isolates M from closed invariant sets and V be a neighborhood of M with $\overline{V} \subset U$. To show that M is positively stable, assume, on the contrary, that M is not positively stable. Then $D^+(M) \neq M$ and so, there is a point x in $D^+(y) \setminus M$ for some y in M. We may assume that x is not in U. Then there is a sequence of points $\{x_i\}$ and a sequence of numbers $\{t_i\}$ in R^+ such that $x_i \to y \in \partial M$ and $x_i t_i \to x$. For each integer i, define $r_i = \inf\{t > 0 | x_i t \in \partial\overline{V}\}$. Since M is positively invariant, we may assume that $x_i[0,\ r_i]$ is contained in the compact set $\overline{V} \setminus M^\circ$ and $r_i \to \infty$. Let $x_i r_i$ converge to z in $\partial\overline{V}$. Suppose $z(-s) \notin \overline{V}$ for a positive number s. Then, for sufficiently large k, we get $0 < r_k - s < r_k$ and $(x_k r_k)(-s) \notin \overline{V}$. This contradicts the fact that $(x_k r_k)(-s) = x_k(r_k - s) \in x_k[0,\ r_k] \subset \overline{V}$. This shows that $\overline{O^-(z)} \subset \overline{V} \setminus M^\circ$. Hence $\alpha(z)$ is nonempty and is contained in $\overline{V} \setminus M^\circ$. By assumtion, $\alpha(z)$ is contained in M and we conclude that Zubov's condition does not hold. This contradiction shows that M is positively stable.

Next, let M be positively stable, then M is positively invariant and $D^-(X \setminus M) \cap M$ is empty [1]. Since $\alpha^-(X \setminus M) \cap M \subset D^-(X \setminus M) \cap M$, M satisfies Zubov's condition. This completes the proof of this result. $\square$

A weak attractor $M \subset X$ is called recursive provided the positive semi-orbit of x is frequently contained in M for each point x of $A^+_W(M)$.

For a point x in M the number of x leaves in M is the cardinality of the class of components of $\{t : xt \in O(x) \setminus M\}$.

THEOREM 1.5. *Let M be a closed recursive positive weak attractor with compact boundary and let the boundary of M contain no positive semi-orbit. Then, M is a saddle set if and only if some point in M leaves M at least twice.*

Proof. Suppose that M is a saddle set for which each point in M leaves M at most once. Then, there is a neighborhood U of M contained in the positive region of weak attraction of M so that $\overline{U} \setminus M^\circ$ is compact. Also, there is a sequence of points $\{x_i\}$ in $U \setminus M$ converging to a point x in ∂M such that $O^+(x_i) \cap \partial U \neq \emptyset$ and $O^-(x_i) \cap \partial U \neq \emptyset$ for each i. Here, we can choose sequences of numbers $\{t_i\} \subset R^+$ and $\{s_i\} \subset R^-$ satisfying that $t_i = \inf\ \{t > 0 : x_i t \in \partial U\}$ and $t_i = \sup\ \{s < 0 : x_i s \in \partial U\}$. Then, cleary, $x_i t_i$ and $x_i s_i$ is in ∂U. Note that $x_i[0,\ t_i] \cap M$ and $x_i[s_i,\ 0] \cap M$ are empty for each i because if $x_i t \in M$ for some t with $0 < t \leq t_i$, then, the point xt in M leaves M at least twice and this is absurd. For any point y in $A^+_W(M)$, since M is a positive recursive weak attractor, there is a number $r > 0$ such that yr in M. If the point yr is in the boundary of M then, the positive semi-orbit of yr is not contained in the boundary of M. If $O^+(yr) \cap M^\circ$ is empty, then $O^+(yr) \cap (X \setminus M) \neq \emptyset$ and this means that the point yr in M leaves M at least twice, but this is absurd. From this, we conclude that for each point y in $A^+_W(M)$, the positive semi-orbit of y must intersect the interior of M. Therefore, for each point y in $A^+_W(M)$, the number $T_y = \inf\{t \geq 0 : yt \in M^\circ\}$ can be defined. Also, define $T = \sup\{t_y : y \in \overline{U} \setminus M^\circ\}$. We shall show that $T < \infty$. For every point y in $\overline{U} - M^\circ$ there exists a neighborhood V_y of y such that $V_y t \subset M^\circ$ for some t between T_y and $T_y + 1$ by the continuity of π. Note that, for any point z of V_y, $T_z \leq T_y + 1$ holds. The set $\{V_y : y \in \overline{U} \setminus M^\circ\}$ is an open covering of the compact set $\overline{U} \setminus M^\circ$ and hence contains a finite subcovering $\{V_{y_1},\ V_{y_2},\ \cdots,\ V_{y_n}\}$ of $U \setminus M^\circ$. Therefore, $T \leq \max\{t_{y_i} + 1 : 1 \leq i \leq n\}$ and thus we get $T < \infty$. By the choice of $\{t_i\}$, we can assume that $0 < t_i \leq T$ for every positive integer i. So, by the continuity of π, we can assume without loss of generality that $x_i t_i \to x\tau_1 \in O^+(x) \cap \partial U$ for some positive integer τ_1.

On the other hand, Since x leaves M at most once, the negative orbit of x is contained in M and thus we may assume that $s_i \to -\infty$. Let $x_i s_i \to x \in \partial U$. By the above statement, $z\tau_2$ is in the interior of M with $0 < \tau_2 < T$. Thus, there exists a positive integer k satisfying that $s_k + \tau_2 < 0$ and $x_k(\tau_2 + s_k)$ is in the interior of M by the continuity of π. This shows that the point $x_t(\tau_2 + s_k)$ leaves M at least twice, which is absurd. Consequently, some point in M leaves M at least twice. The converse is trivial and this completes the proof of this result. □

Let M and Q be a subset of X. M is called a saddle set relative to the set Q if there exists a neighborhood U of M such that every neighborhood V of M contains at least one point x in $V \cap Q$ with $O^+(x) \not\subset U$ and $Q^-(x) \not\subset U$. $M \subset X$ is called a component-wise saddle set if M is a saddle set relative to at least one connected component of $X \setminus M$. The following is trivial.

THEOREM 1.6. *If $M \subset X$ is a component-wise saddle set, then M is a saddle set.*

The following example shows that the converse of the above result does not hold in general.

EXAMPLE 1.7. *Let*

$$X = \{(x, y) \in R^2 : y = 0 \ \text{ or } \ y = \frac{1}{n} \quad \text{for positive integers} \ \ n\}.$$

Consider the flow on X defined by the differential equation

$$x' = y, \quad y' = 0$$

Then, the set $\{(0,0)\} \subset X$ is saddle set but is not a component-wise saddle set.

THEOREM 1.8. *Let $M \subset X$ be a closed invariant saddle set with compact boundary and $X \setminus M$ be locally connected. Then, M is a component-wise saddle set.*

Proof. Since M is a saddle set, there exist a neighborhood U of M with compact boundary, a sequences of points $\{x_n\} \subset U \setminus M$, sequences of numbers $\{t_n\} \subset R^+$, $\{s_n\} \subset R^-$ such that $x_n \to x \in M$, $x_n t_n \in \partial U$ and $x_n s_n \in \partial U$. Let $\{C_\lambda\}$ be the class of connected components of $X \setminus M$. First, we claim that the only finite number of connected components in $\{C_\lambda\}$ can intersect ∂U. Assume, on the contrary, that infinite number of connected components in $\{C_\lambda\}$ intersect ∂U. Then, we can choose a sequence of points $\{y_n\} \subset \partial U$ such that any two points in $\{y_n\}$ does not exist in the same connected components in $\{C_\lambda\}$. Let $y_n \to y \in \partial U$. Since $X \setminus M$ is locally connected, there is a connected neighborhood W of y contained in $X \setminus M$. Then, W is contained in one connected component, namely $C_\alpha \in \{C_\lambda\}$ and there is a positive integer N such that every point y_i, $i \geq N$ is in C_α. But this is absurd. Hence, we conclude that only the finite number of connected components of $X \setminus M$ intersect ∂U.

Let $C_1, C_2, \cdots, C_p$ be connected components of $X \setminus M$ which intersect ∂U. Note that every connected component set is invarinat. Therefore, we can choose a connected component C_k in $\{C_1, C_2, \cdots, C_q\}$, a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\{x_{n_k}\} \subset C_k$. This implies that, for every neighborhood V of M, there is a point $z \in \{x_{n_k}\}$ with $z \in V \cap C_k$ satisfying that $O^+(z) \not\subset U$ and $O^-(x) \not\subset U$. This shows that M is a component-wise saddle set relative to C_k and this completes the proof. $\square$

References

[1] N. Aoki, *Topological Dynamics,* Elsevier Sci. Pub., Tokyo, (1989).

[2] N. P. Bhatia and G. Szego, *Dynamical Systems:Stability Theory and Applications,* Springer-Verlag, Berlin, (1967).

[3] R. Knight, *Unstable weak Attractor,* Proc. Math. Soc. **86** (1982), 586-590.

[4] K. Koo, *On the flows near closed sets,* Natural Science **23** (2012), 169-173.

*

Department of Computer and Information Security
Daejeon University
Daejeon 300-716, Republic of Korea
E-mail: kskoo@dju.kr